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THE CALCULATION OF SEARCH AREAS RELATING TO D.F. FIXES. [U]

BY

BERYL KITZ

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SUMMARY

Formulae are deduced to enable the dimensions of a rectangular region which has a 90% probability that the target lies inside the region, to be calculated. For the special case in which N stations are equi-spaced along an arc at a constant distance, θ, from the target and have the same variance, V, the length and breadth of the search area are proportional to sin θ and $\sqrt{V}$.

If ϕ is the angle subtended by the wing stations at the target then the length of the search region is proportional to

$$F(N,\phi) = \left[N - \left| \frac{\sin (N \phi / N-1)}{\sin (\phi / N-1} \right| \right]^{-\frac{1}{2}}$$

If $\phi < 90^\circ$, the length of the search region is not decreased when the number of stations is increased from 2 to 3; $F(N,\phi)$ decreases as N increases for N greater than 3 . For small values of ϕ

$$F(N,\phi) \doteq \frac{\sqrt{6}}{\phi} \left[\frac{N-1}{N(N+1)} \right]^{\frac{1}{2}}$$

Small values of ϕ should thus be avoided if possible.

The width of the search region is proportional to

$$G(N,\phi) = \left[N + \left| \frac{\sin (N \phi / N-1)}{\sin (\phi / N-1)} \right| \right]^{-\frac{1}{2}}$$

This function decreases as N increases for $\phi < 120^\circ$; if $\phi \geq 120^\circ$ $G(N,\phi)$ decreases as N increases if $N > 3$ and $G(2,\phi) = G(3,\phi)$.

The search region of least area and also the one with the smallest major axis for constant N occurs when $\phi = \frac{(N-1)180^\circ}{N}$

1. INTRODUCTION

When bearings on a target are taken from a number of stations it is not sufficient to plot the bearings and estimate the Best Point for the target. It is common practice to calculate the dimensions of a rectangular region which has a 90% probability that the target lies inside it, in order to estimate the precision of the fix. Various methods have been produced for doing this (cf References 1, 2 and 3) and the method described in References 1 and 2 has been programmed for a number of computers. The method is extremely tedious to compute by hand and it is not clear from the formulae employed how the various parameters of the fix, namely, the number of stations, the variance of the bearings, the distance from the station and the angle subtended by the wing stations at the target, affect the size and shape of the region.

It is important, when the disposition of the stations and the requirements for new apparatus are being studied to consider the way in which various factors affect the size and shape of the region, since this vitally affects the performance of the system in practice. In this report formulae are produced to enable the size and shape of the search region for certain theoretical arrangements to be quickly deduced and the way in which each parameter affects the dimensions of this region is discussed. A comparison of the results obtained using the formulae given in this report with those obtained using the method described in References 1 and 2 is given in the Appendix.

2. DERIVATION OF THE FUNDAMENTAL FORMULAE

The form of the search area under consideration is a band of width $W/2$ on either side of the arc of the great circle passing through the estimated position of the target and joining two points S_1 and S_2. The arc $S_1 S_2$ is called the major axis of the search area and the points S_1 and S_2 will be referred to as its "end points". In order to specify the search area completely it is necessary to find the angle which the major axis makes with the northerly direction at the estimated position of the target, the distances of S_1 and S_2 from this point and the half-width $W/2$.

Let bearings be given from a number of stations A_n.

Suppose that $\delta_s(n)$ is the angle which the bearing from the station A_n makes with the great circle A_nS; $\delta_s(n)$ is called the angular error of the station A_n at the point S. Suppose also that the station A_n has variance V_n measured in $(\text{degrees})^2$. The sum

$$\Sigma_s = \sum_n \delta_s^2(n)/V_n$$

is called the weighted sum of the angular errors at the point S and has a value for every point S on the earth's surface. The estimated position of the target is the point P at which the weighted sum of the angular errors is a minimum, the major axis of the required region on the sphere is the 'major axis' of the contour on the sphere for which

$$\Sigma_s = \Sigma_p + 4\pi^2/180^2$$

and the width of the region is determined as stated below. These definitions are in accordance with those used in References 1, 2 and 3.

It is therefore necessary to find the function Σ_s. When this is known the form of the contour

$$\Sigma_s = \Sigma_p + 4\pi^2/180^2$$

can be ascertained and the direction of its 'major axis' found. The points S_1 and S_2 are points on this great circle for which

$$\Sigma_s = \Sigma_p + 4\pi^2/180^2$$

and thus the length of the search area can be computed. Similarly, the width of the search area can be calculated by finding two points S_3 and S_4 on the great circle which is orthogonal to the major axis at the point P,

and for which

$$\Sigma_s = \Sigma_p + 4\,\pi^2/180^2 \qquad \text{also holds.}$$

In Fig. 1 let P be the estimated position of the target, N be the North Pole (it is assumed that the target lies in the northern hemisphere) and A be a typical station contributing to the fix. Let the bearing from A meet the great circle through P and N in C and let S be a point such that the great circle SP makes an angle ψ with the great circle PN and the angular distance SP = l . Let the great circle through P which meets the great circle AC at right angles meet AC in D .

Let angular distance AP = θ

" " AS = θ_s

" " CP = x

" " DP = d

Let angle ACP = β

angle CAP = δ_p

angle CAS = δ

Using the sine formula in spherical triangle ACS we have

$$\sin \delta = \frac{\sin A\hat{C}S \sin CS}{\sin \theta_s} \tag{1}$$

Using the sine and cosine formulae in spherical triangle CSP we obtain

$$\cos CS = \cos x \cos l + \sin x \sin l \cos (180 - \psi) \tag{2}$$

$$\sin S\hat{C}P = \frac{\sin l \sin \psi}{\sin CS} \tag{3}$$

and

$$\cos S\hat{C}P = \frac{\cos l - \cos x \cos CS}{\sin x \sin CS}$$

$$= \frac{\cos l \sin x + \cos x \sin l \cos \psi}{\sin CS}$$

using (2) (4)

Now $\sin A\hat{C}S = \sin (\beta - S\hat{C}P)$

$$= \sin \beta \cos S\hat{C}P - \cos \beta \sin S\hat{C}P$$

Thus using equations (3) and (4) we get

$$\sin A\hat{C}S \sin CS = \sin l \,[\sin \beta \cos x \cos \psi - \cos \beta \sin \psi] + \cos l \sin \beta \sin x \tag{5}$$

Using the cosine formula in spherical triangle APS we have

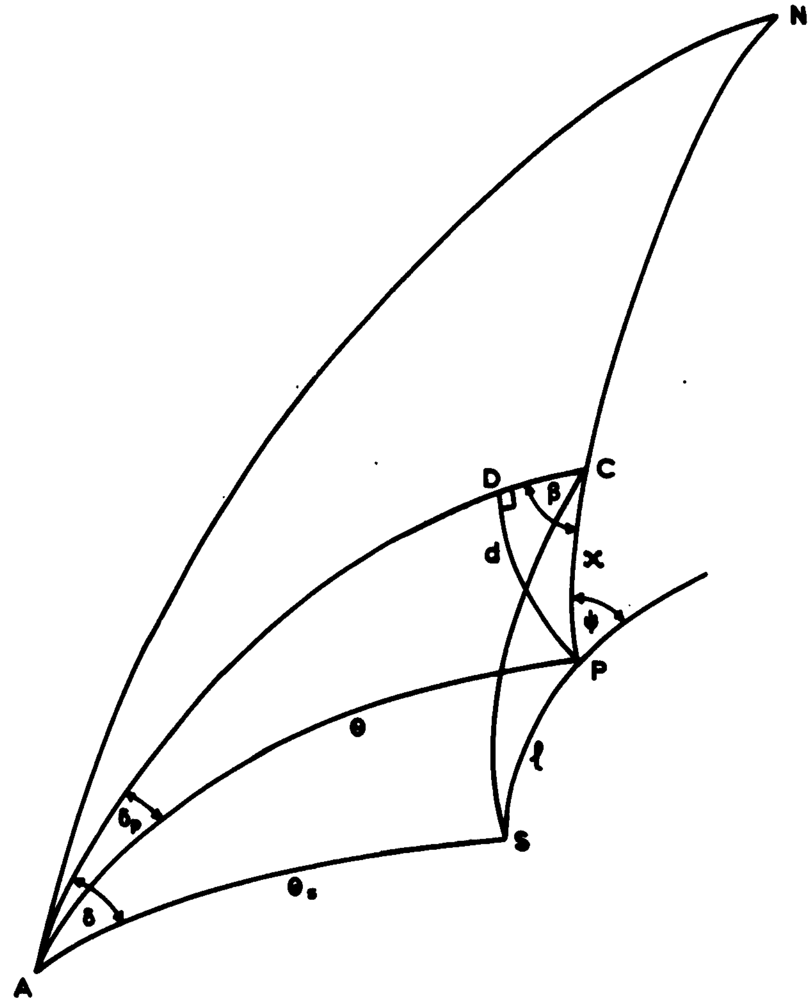

FIG. 1

$$\cos\theta_s = \cos\theta\cos l + \sin\theta\sin l\cos \hat{APS}$$

$$\therefore \quad \sin^2\theta_s = 1 - (\cos\theta\cos l + \sin\theta\sin l\cos \hat{APS})^2$$

$$= \sin^2\theta\,[1-2\cot\theta\sin l\cos l\cos \hat{APS} + \sin^2 l\,(\cot^2\theta - \cos^2 \hat{APS})] \qquad (6)$$

From equations (1), (5) and (6) we find

$$\sin^2\delta = \frac{\left[\sin l\,(\sin\beta\cos\varpi\cos\psi - \cos\beta\sin\psi) + \cos l\sin\beta\sin\varpi\right]^2}{\sin^2\theta\,[1-2\cot\theta\sin l\cos l\cos \hat{APS} + \sin^2 l\,(\cot^2\theta - \cos^2 \hat{APS})]}$$

$$= \frac{\cos^2 l\sin^2\beta\sin^2\varpi + 2\sin l\cos l\sin\beta\sin\varpi\,(\sin\beta\cos\varpi\cos\psi - \cos\beta\sin\psi)}{\sin^2\theta\,[1-2\cot\theta\sin l\cos l\cos \hat{APS} + \sin^2 l\,(\cot^2\theta - \cos^2 \hat{APS})]}$$

$$+ \frac{\sin^2 l\,(\sin^2\beta\cos^2\varpi\cos^2\psi + \sin^2\psi\cos^2\beta - 2\cos\varpi\sin\beta\cos\beta\sin\psi\cos\psi)}{\sin^2\theta\,[1-2\cot\theta\sin l\cos l\cos \hat{APS} + \sin^2 l\,(\cot^2\theta - \cos^2 \hat{APS})]} \qquad (7)$$

If l is small, we may replace $\sin l$ by l and $\cos l$ by $(1-l^2)$ on the right-hand side and obtain

$$\sin^2\delta \doteq \frac{(1-l^2)\sin^2\beta\sin^2\varpi + 2\,l\sin\beta\sin\varpi\,(\sin\beta\cos\varpi\cos\psi - \cos\beta\sin\psi)}{\sin^2\theta\,[1-2\,l\cot\theta\cos \hat{APS} + l^2\,(\cot^2\theta - \cos^2 \hat{APS})]}$$

$$+ \frac{l^2\,(\sin^2\beta\cos^2\varpi\cos^2\psi + \sin^2\psi\cos^2\beta - 2\cos\varpi\sin\beta\cos\beta\sin\psi\cos\psi)}{\sin^2\theta\,[1-2\,l\cot\theta\cos \hat{APS} + l^2\,(\cot^2\theta - \cos^2 \hat{APS})]}$$

$$\doteq \frac{\sin^2\beta\sin^2\varpi}{\sin^2\theta}$$

$$+ \frac{2l}{\sin^2\theta}\Big[\sin\beta\sin\varpi\,(\sin\beta\cos\varpi\cos\psi - \cos\beta\sin\psi) + \sin^2\beta\sin^2\varpi\cot\theta\cos \hat{APS}\Big]$$

$$+ \frac{l^2}{\sin^2\theta}\Big[\sin^2\beta\cos^2\varpi\cos^2\psi + \sin^2\psi\cos^2\beta - 2\cos\varpi\sin\beta\cos\beta\sin\psi\cos\psi - \sin^2\beta\sin^2\varpi$$

$$- \sin^2\beta\sin^2\varpi\,(\cot^2\theta - \cos^2 \hat{APS})$$

$$+ 4\cot\theta\cos \hat{APS}\sin\beta\sin\varpi\,(\sin\beta\cos\varpi\cos\psi - \cos\beta\sin\psi)$$

$$+ 4\cot^2\theta\cos^2 \hat{APS}\sin^2\beta\sin^2\varpi\Big] \qquad (8)$$

Using the sine formula in spherical triangle DCP we have

$$\sin d = \frac{\sin \varpi \sin \beta}{\sin 90^\circ} = \sin \varpi \sin \beta$$

Substituting this in equation (8) we get

$$\sin^2 \delta \doteq \frac{\sin^2 d}{\sin^2 \theta}$$

$$+ \frac{2l}{\sin^2 \theta} \cdot \left[\sin d \, (\sin \beta \cos \varpi \cos \psi - \cos \beta \sin \psi) + \sin^2 d \cot \theta \cos \hat{APS} \right]$$

$$+ \frac{l^2}{\sin^2 \theta} \Big[\sin^2 \beta \cos^2 \varpi \cos^2 \psi + \sin^2 \psi \cos^2 \beta - 2 \cos \varpi \sin \beta \cos \beta \sin \psi \cos \psi$$

$$- \sin^2 d \, (1 + \cot^2 \theta - \cos^2 \hat{APS})$$

$$+ 4 \sin d \cot \theta \cos \hat{APS} \, (\sin \beta \cos \varpi \cos \psi - \cos \beta \sin \psi)$$

$$+ 4 \sin^2 d \cot^2 \theta \cos^2 \hat{APS} \Big] \qquad (10)$$

If the bearing from A is not a wild bearing (which must be excluded from the fix) δ, d and ϖ^2 will be small. Expanding both sides of equation (10) in powers of δ, d and ϖ^2 we obtain, to the second order of small quantities,

$$\delta^2 = \frac{d^2}{\sin^2 \theta} + \frac{2ld}{\sin^2 \theta} (\sin \beta \cos \psi - \cos \beta \sin \psi)$$

$$+ \frac{l^2}{\sin^2 \theta} \left[\sin^2 \beta \cos^2 \psi + \sin^2 \psi \cos^2 \beta - 2 \sin \beta \cos \beta \sin \psi \cos \psi \right]$$

Suppose that N stations contribute to the fix under consideration and that the parameters defined above with suffix n refer to the nth station and that this station has variance V_n. The weighted sum of the angular errors at S is given by

$$\Sigma_s = \sum_{n=1}^{N} \frac{\delta_n^2}{V_n}$$

$$\doteq \sum_{n=1}^{N} \frac{d_n^2}{V_n \sin^2 \theta_n} + 2l \left[\cos \psi \sum_{n=1}^{N} \frac{d_n \sin \beta_n}{V_n \sin^2 \theta_n} - \sin \psi \sum_{n=1}^{N} \frac{d_n \cos \beta_n}{V_n \sin^2 \theta_n} \right]$$

$$+ l^2 \left[\cos^2 \psi \sum_{n=1}^{N} \frac{\sin^2 \beta_n}{V_n \sin^2 \theta_n} + \sin^2 \psi \sum_{n=1}^{N} \frac{\cos^2 \beta_n}{V_n \sin^2 \theta_n} - 2 \sin \psi \cos \psi \sum_{n=1}^{N} \frac{\sin \beta_n \cos \beta_n}{V_n \sin^2 \theta_n} \right]$$

(11)

Using the sine rule in spherical triangle $\hat{ACP}$ we have

$$\sin \delta_p = \frac{\sin s \sin \beta}{\sin \theta} = \frac{\sin d}{\sin \theta}$$

which becomes, if we ignore terms of order higher than δ_p^2 and d^2

$$\delta_p \doteq \frac{d}{\sin \theta}$$

and thus the weighted sum of the square of the angular errors at the Best Point P is given by

$$\Sigma_p = \sum_{n=1}^{N} \frac{\delta^2 p_n}{V_n} \doteq \sum_{n=1}^{N} \frac{d_n^2}{V_n^2 \sin^2 \theta_n}$$

Since P is the Best Point corresponding to the fix the weighted sum of the square of the angular errors is a minimum there and thus the coefficients of the first power of l in equation (11) must vanish.

We have therefore

$$\Sigma_s = \Sigma_p + l^2 \left[\cos^2 \psi \sum_{n=1}^{N} \frac{\sin^2 \beta_n}{V_n \sin^2 \theta_n} + \sin^2 \psi \sum_{n=1}^{N} \frac{\cos^2 \beta_n}{V_n \sin^2 \theta_n} - 2 \sin \psi \cos \psi \sum_{n=1}^{N} \frac{\sin \beta_n \cos \beta_n}{V_n \sin^2 \theta_n} \right] \tag{12}$$

Since, by Schwarz's inequality

$$\left[\sum_{n=1}^{N} \frac{\sin^2 \beta_n}{V_n \sin^2 \theta_n}\right]\left[\sum_{n=1}^{N} \frac{\cos^2 \beta_n}{V_n \sin^2 \theta_n}\right] - \left[\sum_{n=1}^{N} \frac{\sin \beta_n \cos \beta_n}{V_n \sin^2 \theta_n}\right]^2 > 0$$

unless $\tan \beta_n$ is constant for all the stations, that is, unless all the bearings are parallel when they cross the great circle PN, the curves Σ_s = constant are in general ellipses for small values of l. If $\tan \beta_n$ is constant for all the stations, the curves Σ_s = constant are parabolas. In either case the bearing Ψ of the search area is that of the major axis of these conics and is given by

$$\tan 2 \Psi = \left[- 2 \sum_{n=1}^{N} \frac{\sin \beta_n \cos \beta_n}{V_n \sin^2 \theta_n} \right] \Big/ \left[\sum_{n=1}^{N} \frac{\sin^2 \beta_n}{V_n \sin^2 \theta_n} - \sum_{n=1}^{N} \frac{\cos^2 \beta_n}{V_n \sin^2 \theta_n} \right]$$

$$= \left[+ \sum_{n=1}^{N} \frac{\sin 2 \beta_n}{V_n \sin^2 \theta_n} \right] \Big/ \left[\sum_{n=1}^{N} \frac{\cos 2 \beta_n}{V_n \sin^2 \theta_n} \right] \tag{13}$$

The four end points of the axes of the search region are points at which $\psi = \Psi_r$, where Ψ_r is one of the four angles given by equation (13), and $l = l_r$ where l_r is given by the equation

$$\sum_s - \sum_p = \frac{4\pi^2}{180^2}$$

if the variances are measured in (degrees)^2 .

We thus have

$$l_r^2 \left[\cos^2 \Psi_r \sum_{n=1}^{N} \frac{\sin^2 \beta_n}{V_n \sin^2 \theta_n} - 2 \sin \Psi_r \cos \Psi_r \sum_{n=1}^{N} \frac{\sin \beta_n \cos \beta_n}{V_n \sin^2 \theta_n} \right.$$

$$\left. + \sin^2 \Psi_r \sum_{n=1}^{N} \frac{\cos^2 \beta_n}{V_n \sin^2 \theta_n} \right] = \frac{4\pi^2}{180^2} \quad \text{where } r = 1,2,3,4$$

or

$$l_r^2 \left[\left(\frac{1}{2} + \frac{1}{2} \cos 2 \Psi_r \right) \sum_{n=1}^{N} \frac{\left(\frac{1}{2} - \frac{1}{2} \cos 2 \beta_n \right)}{V_n \sin^2 \theta_n} - \frac{1}{2} \sin 2 \Psi_r \sum_{n=1}^{N} \frac{\sin 2 \beta_n}{V_n \sin^2 \theta_n} \right.$$

$$\left. + \left(\frac{1}{2} - \frac{1}{2} \cos 2 \Psi_r \right) \sum_{n=1}^{N} \frac{\left(\frac{1}{2} + \frac{1}{2} \cos 2 \beta_n \right)}{V_n \sin^2 \theta_n} \right] = \frac{4\pi^2}{180^2} \tag{14}$$

$$l_r^2 \left[\frac{1}{2} \sum_{n=1}^{N} \frac{1}{V_n \sin^2 \theta_n} - \frac{1}{2} \cos 2 \Psi_r \sum_{n=1}^{N} \frac{\cos 2 \beta_n}{V_n \sin^2 \theta_n} \left\{ 1 + \tan 2\Psi_r \frac{\sum_{n=1}^{N} \frac{\sin 2 \beta_n}{V_n \sin^2 \theta_n}}{\sum_{n=1}^{N} \frac{\cos 2 \beta_n}{V_n \sin^2 \theta_n}} \right\} \right]$$

$$= \frac{4\pi^2}{180^2}$$

Thus
$$l_r = \frac{2\sqrt{2}\,\pi/180}{\left[\sum_{n=1}^{N} \frac{1}{V_n \sin^2 \theta_n} - \sec 2 \Psi_r \sum_{n=1}^{N} \frac{\cos 2 \beta_n}{V_n \sin^2 \theta_n} \right]^{\frac{1}{2}}}$$

l_r being measured in radians. The length of the search region L is given by $l_1 + l_2$ where Ψ_1 and Ψ_2 are the values of Ψ_r given by equation (13) for which $\cos 2 \Psi_r$ and

$$\sum_{n=1}^{N} \frac{\cos 2 \beta_n}{V_n \sin^2 \theta_n} \quad \text{have the same sign and thus}$$

$$L \doteq \frac{240\sqrt{2}}{\left[\sum_{n=1}^{N}\frac{1}{V_n \sin^2\theta_n} - \left|\sec 2\Psi \sum_{n=1}^{N}\frac{\cos 2\beta_n}{V_n \sin^2\theta_n}\right|\right]^{\frac{1}{2}}}$$ nautical miles

Similarly the width of the search region, W, is

$$W \doteq \frac{240\sqrt{2}}{\left[\sum_{n=1}^{N}\frac{1}{V_n \sin^2\theta_n} + \left|\sec 2\Psi \sum_{n=1}^{N}\frac{\cos 2\beta_n}{V_n \sin^2\theta_n}\right|\right]^{\frac{1}{2}}}$$ nautical miles

3. DERIVATION OF FORMULAE FOR SYMMETRICAL ARRANGEMENTS OF STATIONS

Consider the theoretical case in which all the stations are equispaced along an arc at the same angular distance θ from the target and have the same variance V. The practical case frequently approximates to this when all the stations are using similar equipment and the target is a long way away. Assume further that all the bearings pass through the point P. The bearing of the search region Ψ is given by the equation,

$$\tan 2\Psi = \left[\sum_{n=1}^{N}\frac{\sin 2\beta_n}{V_n \sin^2\theta_n}\right] \Big/ \left[\sum_{n=1}^{N}\frac{\cos 2\beta_n}{V_n \sin^2\theta_n}\right]$$

$$= \left[\sum_{n=1}^{N}\sin 2\beta_n\right] \Big/ \left[\sum_{n=1}^{N}\cos 2\beta_n\right]$$

and the length and breadth of the search region in nautical miles are given by the formula

$$\frac{240\sqrt{2}}{\left[\sum_{n=1}^{N}\frac{1}{V_n \sin^2\theta_n} \mp \left|\sec 2\Psi \sum_{n=1}^{N}\frac{\cos 2\beta_n}{V_n \sin^2\theta_n}\right|\right]^{\frac{1}{2}}}$$

$$= \frac{240\sqrt{2}\sqrt{V}\sin\theta}{\left[N \mp \left|\sec 2\Psi \sum_{n=1}^{N}\cos 2\beta_n\right|\right]^{\frac{1}{2}}}$$

Suppose that the stations A_1, A_2 .. A_N are arranged as in Fig. 2 so that

$$A_1 \hat{P} A_2 = A_2 \hat{P} A_3 = A_3 \hat{P} A_4 = \ldots\ldots = A_{N-1} \hat{P} A_N = \gamma$$

and that the bearing from A_1 to P makes an angle η with PN. Then

$$\beta_n = 180^\circ - N\hat{P}A_n = 180^\circ - \left(180^\circ - \eta + (n-1)\gamma\right)$$

or $\quad \beta_n = \eta - (n-1)\gamma \qquad\qquad n = 1, 2 .. N$

Thus $$\sum_{n=1}^{N} \sin 2\beta_n = \sum_{n=1}^{N} \sin\left[2\eta - 2(n-1)\gamma\right]$$

$$= \sin\left[2\eta - (N-1)\gamma\right] \sin N\gamma \operatorname{cosec}\gamma$$

and $$\sum_{n=1}^{N} \cos 2\beta_n = \sum_{n=1}^{N} \cos\left[2\eta - 2(n-1)\gamma\right]$$

$$= \cos\left[2\eta - (N-1)\gamma\right] \sin N\gamma \operatorname{cosec}\gamma$$

Thus if $\sin N\gamma \neq 0$ the bearing of the search region is given by the equation

$$\tan 2\Psi = \tan\left[2\eta - (N-1)\gamma\right]$$

and $$|\sec 2\Psi| = |\sec\left[2\eta - (N-1)\gamma\right]|$$

The length and breadth of the search area in nautical miles are thus given by the formula

$$\frac{240\sqrt{2}\sqrt{V}\sin\theta}{\left[N \mp |\sec[2\eta - (n-1)\gamma]\cos[2\eta - (n-1)\gamma]\sin N\gamma \operatorname{cosec}\gamma\right]^{\frac{1}{2}}}$$

$$= \frac{240\sqrt{2}\sqrt{V}\sin\theta}{\left[N \mp |\sin N\gamma \operatorname{cosec}\gamma|\right]^{\frac{1}{2}}}$$

If $A_1 \hat{P} A_N = \phi$ then $\gamma = \phi/N-1$ and the bearings of the axes of the search region are given by

$$\Psi = \eta - \phi/2$$

and $\Psi = \eta - \phi/2 + 90^\circ$

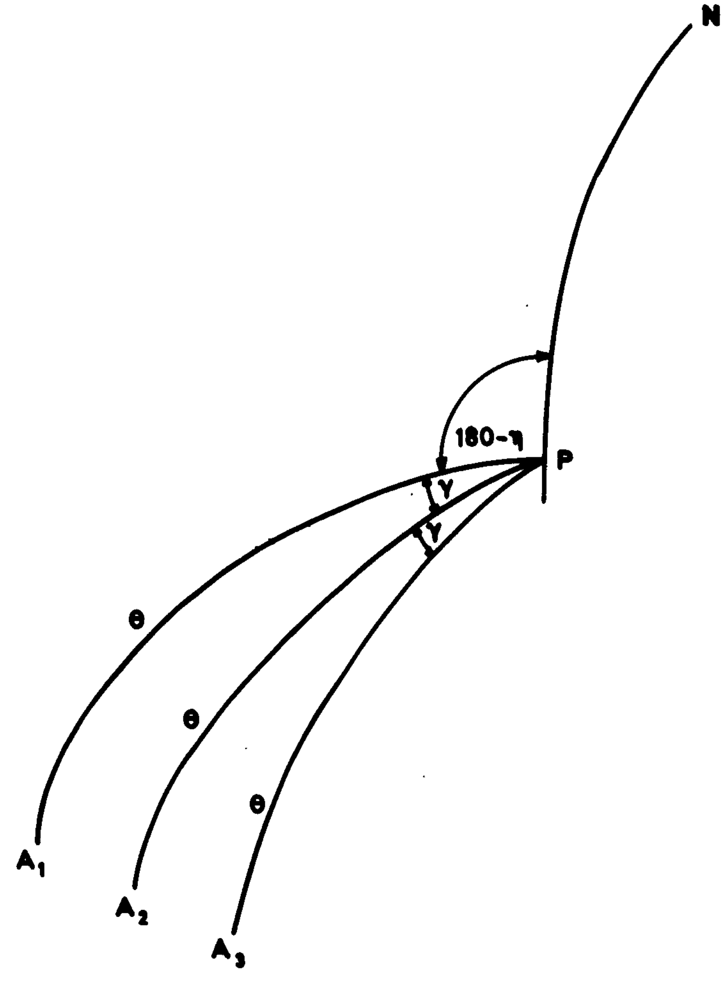

FIG. 2

the bearing of the major axis being that value of Ψ for which $\cos 2\Psi$ and $\cos (2\eta-\phi) \sin (N\phi/N-1) \operatorname{cosec} (\phi/N-1)$ have the same sign. The length and breadth of the search region in nautical miles are given by the formula

$$\frac{240 \sqrt{2} \sqrt{V} \sin \theta}{\left[N \mp \left| \sin (N\phi/N-1)/\sin (\phi/N-1) \right|\right]^{\frac{1}{2}}} \qquad (15)$$

If $\sin N\gamma = 0$ then $\gamma = \kappa\pi/N$. In this case the stations are symmetrically placed with respect to P and the elliptical contours become circles. The length and breadth of the search region may be obtained directly from equation (14) since

$$\sum_{n=1}^{N} \sin 2\beta_n = \sum_{n=1}^{N} \cos 2\beta_n = 0$$

and thus

$$l_r^2\left[\frac{1}{2} \sum_{n=1}^{N} \frac{1}{V_n \sin^2 \theta_n}\right] = 4\pi^2/180^2$$

The length and breadth of the search region are equal, as is apparent from the symmetry of the configuration, and

$$L = W = 240 \sqrt{2} \sqrt{V} \sin \theta/\sqrt{N} \text{ nautical miles.}$$

4. ASSESSMENT OF THE ACCURACY OF THE FORMULAE FOR THE LENGTH AND BREADTH OF THE SEARCH REGION

The formulae derived in the previous sections assume that powers of l higher than l^2, where l is the arc length of the axis under consideration measured in radians, may be neglected. Thus, the longer the axis the greater the error becomes. Since powers of δ, d and ϵ higher than the second and products of these terms are also neglected, the calculated values of the lengths of the axes of the search region are more in error if the fix contains inaccurate bearings. Since in general, a fix containing inaccurate bearings will also have a large search area the length of the axis can be used as a guide in all cases.

A number of calculations have been made comparing the results obtained using formula (15) for stations arranged in the symmetrical way described in section 3 with the results obtained using the analysis described in References 1 and 2. In all cases the values obtained for the width of the search region agreed exactly. When V = 1 (degree)^2 no error in the length of the search region exceeded 2% and when the angle subtended at the target by the wing stations was 40° or more no length differed from that computed by the method described in References 1 and 2 by more than 1 nautical mile. When V = 5 (degrees)^2, the angle subtended at the target by the wing stations = 20°, the distance of the stations from the target = 15° and three stations contribute to the fix there is a discrepancy of 10%. The method used for comparison is not accurate for fixes of this nature which increases the difficulty in assessing the situation. All the computed results are given in the Appendix.

If terms of higher power than l^2 are ignored the resulting equation gives values of l which are equal in modulus but differ in sign for the two ends of the search region. It is known that in fact the estimated position of the target is nearer to the stations than the mid-point of the major axis of the search area. If the position of the end points of the axes of the search region are required, additional powers of l must be included in the expression for Σ_s. Let us assume that

$$\Sigma_s = \Sigma_p + \alpha l^2 + \beta l^3 + \gamma l^4$$

and that

$$L = \frac{2\sqrt{2}\pi/180}{\left[\sum_{n=1}^{N} \frac{1}{V_n \sin^2 \theta_n} - \left| \sec 2\Psi \sum_{n=1}^{N} \frac{\cos 2\phi_n}{V_n \sin^2 \theta_n} \right| \right]^{\frac{1}{2}}}$$

is the first approximation to the value of l for one end point of the search region, obtained by putting β and γ equal to zero. Suppose also that the true value of the weighted sum of the squares of the angular errors obtained by using equation (7), which is exact, to compute δ for each station at the point on the axis of the search region for which $l = L$ is Σ_1 whilst the value of this function at the point on the axis of the search region for which $l = -L$ is Σ_2 . We thus have

$$\alpha L^2 = 4\pi^2/180^2 \qquad \text{from equation (14)}$$

$$\Sigma_1 = \Sigma_p + \alpha L^2 + \beta L^3 + \gamma L^4 \qquad (16)$$

and $$\Sigma_2 = \Sigma_p + \alpha L^2 - \beta L^3 + \gamma L^4 \qquad (17)$$

Adding (16) and (17) we obtain

$$\gamma L^4 = \tfrac{1}{2}\left(\Sigma_1 + \Sigma_2\right) - \Sigma_p - 4\pi^2/180^2$$

and subtracting (17) from (16) we get

$$\beta L^3 = \tfrac{1}{2}\left(\Sigma_1 - \Sigma_2\right)$$

and thus

$$\Sigma_s = \Sigma_p + \frac{4\pi^2}{180^2}\left(\frac{l}{L}\right)^2 + \tfrac{1}{2}\left(\Sigma_1 - \Sigma_2\right)\left(\frac{l}{L}\right)^3 + \left\{\tfrac{1}{2}\left(\Sigma_1 + \Sigma_2\right) - \Sigma_p - \frac{4\pi^2}{180^2}\right\}\left(\frac{l}{L}\right)^4 \qquad (18)$$

A better approximation to the positive value of l for which $\Sigma_s = \Sigma_p + \frac{4\pi^2}{180^2}$

is given by

$$l_1 = L\left[\frac{5\,\Sigma_1 + \Sigma_2 - 6\,\Sigma_p - 8\,\pi^2/180^2}{7\,\Sigma_1 + \Sigma_2 - 8\,\Sigma_p - 16\,\pi^2/180^2}\right]$$

by using Newton's formula. Similarly a better approximation to the negative value of l for which

$$\Sigma_s = \Sigma_p + 4\,\pi^2/180^2$$

is given by

$$l_2 = -L\left[\frac{5\,\Sigma_2 + \Sigma_1 - 6\,\Sigma_p - 8\,\pi^2/180^2}{7\,\Sigma_2 + \Sigma_1 - 8\,\Sigma_p - 16\,\pi^2/180^2}\right]$$

The accuracy of the new values of l_1 and l_2 should be tested by substituting them in equation (18) and checking that Σ_s is equal to $\Sigma_p + \dfrac{4\,\pi^2}{180^2}$.

The values of l_1 and l_2 may be improved by further application of Newton's formula using the equation

$$l_1' = l_1 - \frac{\left(\alpha l_1^2 + \beta l_1^3 + \gamma l_1^4 - 4\,\pi^2/180^2\right)}{\left(2\,\alpha l_1 + 3\,\beta l_1^2 + 4\,\gamma l_1^3\right)}$$

As a final check the value of Σ_s at the end points should be computed using equation (7) . If these are not satisfactory more terms must be taken in the series for Σ_s and a procedure similar to that outlined above adopted.

5. THE EFFECT OF THE PARAMETERS OF A FIX ON THE SIZE OF THE SEARCH REGION

It will be seen from the Appendix that formula (15) is sufficiently accurate to enable it to be used to study the way in which the various parameters of a fix namely, the number of stations, the variance of the stations, the distance from the target and the angle which the wing stations subtend at the target affect the size of the search area. A few salient points are given below:-

5.1. The length and breadth of the search area are both proportional to the sine of the angular distance of the stations from the target. For small distances the dimensions of the search area are proportional to the distance from the target.

5.2. The length and breadth of the search area are both proportional to the square root of the variance of the stations. If the variance is not known precisely, as is usually the case since it often depends on physical factors such as propagation conditions which cannot be controlled, then the percentage error in the dimensions of the search area due to this inexactness is approximately half the percentage error in the variance. In practice it is not usually possible to obtain the dimensions of the search area correct to more than two significant figures; this corresponds to a 2% error in estimating the variance.

5.3. The length of the search area is proportional to

$$F(N,\phi) = \left[N - \left| \frac{\sin N\phi/N-1}{\sin \phi/N-1} \right| \right]^{-\frac{1}{2}}$$

where N is the number of stations and ϕ is the angle subtended at the target by the wing stations. If $\phi < 90^\circ$

$$F(2,\phi) = \left[2 - \sin 2\phi / \sin \phi \right]^{-\frac{1}{2}} = \tfrac{1}{2} \operatorname{cosec} \phi/2$$

$$F(3,\phi) = \left[3 - \sin 3\phi/2 \Big/ \sin \phi/2 \right]^{-\frac{1}{2}} = \tfrac{1}{2} \operatorname{cosec} \phi/2$$

Thus, if all the other parameters of the system remain the same, the length of the search area is not reduced by increasing the number of stations from 2 to 3. It is obvious from physical considerations that this must be the case since, if the system is symmetrical about the middle bearing, the axis of the search area must lie along it and the middle bearing therefore makes no contribution to the weighted sum of the angular errors at any point along the major axis of the search area. For values of N greater than 3 the function $F(N,\phi)$ decreases as N increases provided that ϕ remains constant. This is shown in the table of the dimensions of search areas for various fixes given in the Appendix.

5.4. The width of the search area is proportional to

$$G(N,\phi) = \left[N + \left| \frac{\sin N\phi/N-1}{\sin \phi/N-1} \right| \right]^{-\frac{1}{2}}$$

This function decreases as N increases for values of $\phi < 120^\circ$. If $180^\circ > \phi > 120^\circ$, $G(2,\phi) = G(3,\phi)$

5.5. For small values of ϕ

$$F(N,\phi) = \left[N - \frac{\{N\phi/N-1 - \frac{1}{6}(N\phi/N-1)^3\}}{\{\phi/N-1 - \frac{1}{6}(\phi/N-1)^3\}} \right]^{-\frac{1}{2}} \text{ to } 0(\phi^2)$$

$$\doteq \frac{\sqrt{6}}{\phi} \left[\frac{N-1}{N(N+1)} \right]^{\frac{1}{2}}$$

Thus $F(N,\phi)$ increases as $1/\phi$ as ϕ tends to zero. For this reason a small value of ϕ (which is equivalent to a narrow base line) should be avoided if possible since it is not easy to mitigate its effects by varying the other parameters of the system. It is necessary to increase N from 2 to 22, or to

reduce the variance to one quarter of its original value or to halve the distance to the target to achieve the same effect as that obtained by doubling ϕ, that is, by doubling the base line, if ϕ is small. It will be seen from the Appendix that the formula for the length of the search area is not very accurate when ϕ is small and that the situation is even worse than that predicted from formula (15). It is therefore of great importance to extend the base line as far as possible if ϕ is likely to be small if the best possible results are to be obtained.

5.6. The search area of least area and also the one with the smallest major axis for a given number of stations occurs when $\phi = (N-1)\ 180^{\circ}/N$. This arrangement gives the best results for a given amount of equipment but the length of the resulting base line is usually greater than can be tolerated in practice.

B. Kits (P.S.O. actg.)
A.R.L., Teddington
BK/MM

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APPENDIX

In the following tables let

θ	=	angular distance of the stations from the target
ϕ	=	angle subtended at the target by the wing stations
N	=	number of stations
V	=	variance of the bearings
L_1	=	length of search area as calculated by formula (15)
L_2	=	length of search area as calculated by the method described in References 1 and 2
W_1	=	width of search area as calculated by formula (15)
W_2	=	width of search area as calculated by the method described in References 1 and 2

V	N	θ	ϕ	L_1	L_2	W_1	W_2
1	2	15°	20°	253	258	45	45
1	2	15°	40°	128	129	47	47
1	2	15°	60°	88	88	51	51
1	2	30°	20°	489	495	86	86
1	2	30°	40°	248	249	90	90
1	2	30°	60°	170	170	98	98
1	2	30°	80°	132	132	111	111
1	3	15°	20°	253	258	36	36
1	3	15°	40°	128	129	37	37
1	3	15°	60°	88	88	39	39
1	3	15°	80°	68	68	42	42
1	3	15°	100°	57	57	46	46
1	3	15°	120°	51	51	51	51
1	3	15°	140°	56	56	47	47
1	3	15°	160°	60	60	45	45
1	3	15°	180°	62	62	44	44
1	3	30°	20°	489	495	70	70
1	3	30°	40°	248	249	72	72
1	3	30°	60°	170	170	76	76
1	3	30°	80°	132	132	81	81
1	3	30°	100°	111	111	89	89
1	3	30°	120°	98	98	98	98
1	3	30°	140°	108	108	90	90
1	3	30°	160°	117	117	86	86
1	3	30°	180°	120	120	85	85
1	3	45°	20°	691	696	99	99
1	3	45°	40°	351	351	102	102
1	3	45°	60°	240	240	107	107
1	3	45°	80°	187	187	115	115
1	3	45°	100°	157	157	126	126
1	3	45°	120°	139	139	139	139
1	3	45°	140°	153	153	128	128
1	3	45°	160°	165	165	122	122
1	3	45°	180°	170	170	120	120
1	3	60°	20°	846	846	121	121
1	3	60°	40°	430	430	125	125
1	3	60°	60°	294	294	131	131
1	3	60°	80°	229	229	141	141

1	3	60°	100°	192	192	158	158
1	3	60°	120°	170	170	170	170
1	3	60°	140°	187	187	156	156
1	3	60°	160°	202	202	149	149
1	3	60°	180°	208	208	147	147
1	5	15°	20°	226	229	28	28
1	5	15°	40°	115	115	29	29
1	5	15°	60°	78	78	30	30
1	5	15°	80°	60	60	31	31
1	5	15°	100°	50	50	33	32
1	5	15°	120°	44	44	36	36
1	5	15°	140°	40	40	39	39
1	5	15°	160°	42	42	37	37
1	5	15°	180°	44	44	36	36
1	5	30°	20°	437	444	54	54
1	5	30°	40°	221	222	55	55
1	5	30°	60°	154	154	57	57
1	5	30°	80°	147	147	60	60
1	5	30°	100°	97	97	64	64
1	5	30°	120°	85	85	68	68
1	5	30°	140°	77	77	75	75
1	5	30°	160°	80	80	72	72
1	5	30°	180°	85	85	69	69
1	11	15°	20°	170	172	19	19
1	11	15°	40°	[illegible]	[illegible]	19	19
1	11	15°	60°	58	58	20	20
1	11	15°	80°	45	45	21	21
1	11	15°	100°	37	37	22	22
1	11	15°	120°	32	32	25	25
1	11	15°	140°	29	29	25	25
1	11	15°	160°	27	27	24	24
1	11	15°	180°	28	28	25	25
1	11	30°	20°	329	331	36	36
1	11	30°	40°	166	167	37	37
1	11	30°	60°	113	113	38	38
1	11	30°	80°	87	87	40	40
1	11	30°	100°	72	72	42	42
1	11	30°	120°	62	62	44	44
1	11	30°	140°	56	56	47	47
1	11	30°	160°	52	52	50	50

V	N	θ	ϕ	L_1	L_2	W_1	W_2
1	11	30°	180°	54	54	49	49
5	3	15°	20°	566	621	81	81
5	3	15°	40°	287	294	84	84
5	3	15°	60°	196	198	88	88
5	3	15°	80°	153	154	94	94
5	3	15°	100°	128	129	103	103
5	3	15°	120°	113	113	113	113
5	3	30°	20°	1092	1167	157	157
5	3	30°	40°	555	564	161	161
5	3	30°	60°	380	382	170	170
5	3	30°	80°	295	296	182	182
5	3	30°	100°	248	248	199	199
5	3	30°	120°	219	219	219	219
5	3	45°	20°	1545	1593	221	221
5	3	45°	40°	785	791	228	228
5	3	45°	60°	537	538	240	240
5	3	45°	80°	417	418	257	257
5	3	45°	100°	350	351	281	281
5	3	45°	120°	310	310	310	310
5	3	60°	20°	1892	1888	271	271
5	3	60°	40°	961	960	279	279
5	3	60°	60°	657	657	294	294
5	3	60°	80°	511	511	315	315
5	3	60°	100°	429	429	344	344
5	3	60°	120°	379	379	379	379
5	4	15°	60°	186	187	75	75
5	5	15°	60°	174	176	66	66
5	6	15°	60°	165	166	60	60
5	4	30°	60°	359	361	145	145
5	5	30°	60°	337	339	128	128
5	6	30°	60°	318	319	117	117
5	4	45°	60°	507	508	205	205
5	5	45°	60°	477	478	182	182
5	6	45°	60°	450	451	165	165
5	4	60°	60°	621	621	251	251
5	5	60°	60°	584	584	222	222
5	6	60°	60°	550	550	202	202

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